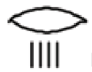
Egyptian Fractions

Fractions have a long history of use in mathematics, but they have not always been written as we see them today. About 5,000 years ago the ancient Egyptians represented fractions using symbols like these: 

The  shape means part, and the marks indicate the parts of the whole, so  represents $\frac{1}{3}$.

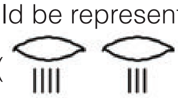
1. What fraction do you think that  represents?

2. Using this pattern, how would you represent $\frac{1}{5}$?

For the most part, the Egyptians used only *unit fractions* (a fraction with the number 1 as the numerator). One of the few fractions that existed in a form other than a unit fraction was $\frac{2}{3}$. It was represented as follows: 
Furthermore, fractions were represented ***without repeating the same*** fractions by using sums of progressively smaller fractions. For example, $\frac{5}{9}$ would not be represented as $\frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9}$ but rather as $\frac{1}{2} + \frac{1}{18}$. We call this representation an *Egyptian fraction*.

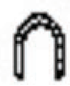
3. What sum of unit fractions could be used to represent $\frac{5}{8}$?

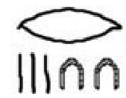
The symbol for $\frac{1}{2}$ is different than the other Egyptian symbols. It is written like this: 
The symbols were written next to each other to show a sum of fractions. For example, $\frac{3}{4}$ would be represented as the sum of $\frac{1}{2}$ and $\frac{1}{4}$. It would be written like this:

4. What fraction could be represented by the two-unit fraction symbols () at the start of this activity?

5. How could you represent $\frac{5}{8}$ using Egyptian symbols?

The Egyptians used different hieroglyphs to represent different values:

1	10	100	1000
			
Single Stroke	Cattle Hobble	Rope Coil	Lotus Plant

They also used these symbols to represent fractions. For example, according to the table above, we can represent $\frac{1}{23}$ in the following manner: 

6. How would you represent $\frac{1}{12}$ using Egyptian hieroglyphs?

7. How would you represent $\frac{1}{1,234}$ using Egyptian hieroglyphs?

A Method to Determine Egyptian Fractions

A useful method for translating a given fraction into an Egyptian fraction is to determine if $\frac{2}{3}$ is less than the given fraction. If it is not, determine the largest unit fraction that is less than the given fraction. List the largest unit fraction (in some cases $\frac{2}{3}$) as the first part of the Egyptian fraction. Then subtract it from the given fraction. Use the result, or the “leftover” fraction, to find the next unit fraction that is smaller than the “leftover” fraction. Repeat this process until the “leftover” fraction is a unit fraction.

For example, in determining the Egyptian fraction for $\frac{5}{6}$, first determine if $\frac{2}{3}$ is less than $\frac{5}{6}$. Since $\frac{2}{3}$ is equivalent to $\frac{4}{6}$, and $\frac{4}{6}$ is less than $\frac{5}{6}$, we know that $\frac{2}{3}$ can be used as the first part of the Egyptian fraction for $\frac{5}{6}$. So, we subtract $\frac{2}{3}$ from $\frac{5}{6}$.

We know that $\frac{5}{6} - \frac{2}{3} = \frac{5}{6} - \frac{4}{6} = \frac{1}{6}$. Since $\frac{1}{6}$ is in the form of a unit fraction, we use the "leftover" of $\frac{1}{6}$ to complete the Egyptian fraction for $\frac{5}{6}$. Therefore, an Egyptian fraction for $\frac{5}{6}$ is $\frac{2}{3} + \frac{1}{6}$. Let's start this process using the fraction $\frac{3}{8}$.

8. Is $\frac{3}{8}$ greater than or less than $\frac{1}{2}$? How do you know?

It is helpful to be able to compare fractions to $\frac{1}{2}$ without finding common denominators. We can compare the numerator of a fraction to its denominator in the following ways:

- If the numerator is less than $\frac{1}{2}$ the denominator, then the fraction is less than $\frac{1}{2}$;
- If the numerator is greater than $\frac{1}{2}$ the denominator, then the fraction is greater than $\frac{1}{2}$.

This is also an efficient strategy for comparing two fractions when one is greater than $\frac{1}{2}$ and the other is less than $\frac{1}{2}$. Finding common denominators becomes unnecessary. Consider $\frac{9}{14}$ and $\frac{5}{12}$. We know that $\frac{9}{14}$ is greater than $\frac{1}{2}$ because 9 is greater than 7 (which is $\frac{1}{2}$ of 14). We also know that $\frac{5}{12}$ is less than $\frac{1}{2}$ because 5 is less than 6 (which is $\frac{1}{2}$ of 12). Therefore, by using the benchmark fraction of $\frac{1}{2}$, we know that $\frac{9}{14} > \frac{5}{12}$.

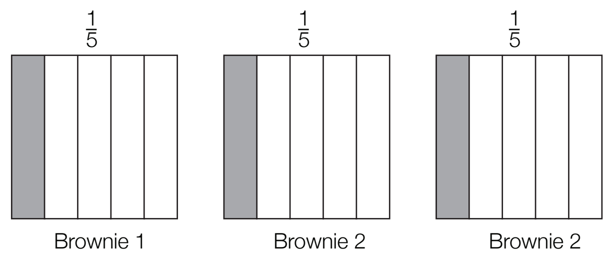
9. Using the benchmark of $\frac{1}{2}$ to compare $\frac{3}{7}$ and $\frac{5}{9}$, determine which fraction is greater. Explain your reasoning.

Let's return to finding an Egyptian fraction to represent $\frac{3}{8}$. We saw that $\frac{3}{8}$ is less than $\frac{1}{2}$. A unit fraction that is less than $\frac{3}{8}$ is $\frac{1}{4}$. List $\frac{1}{4}$ as the first unit fraction in the Egyptian fraction for $\frac{3}{8}$ and then subtract $\frac{1}{4}$ from $\frac{3}{8}$, leaving 1/8. This "leftover" portion of $\frac{3}{8}$ is already in the form of a unit fraction, so the process is complete. An Egyptian fraction for $\frac{3}{8}$ is $\frac{1}{4} + \frac{1}{8}$, or these symbols:.

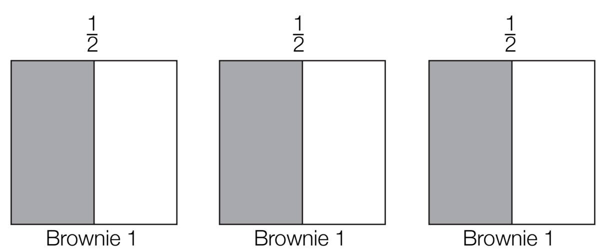
10. Determine an Egyptian fraction for $\frac{5}{9}$.

11. Determine two different ways to represent $\frac{3}{4}$ using Egyptian fractions. (Hint: You are not required to use the largest unit fraction less than $\frac{3}{4}$.)

This method of determining Egyptian fractions has a useful application to sharing situations. Consider sharing 3 brownies among 5 people. We could share the brownies fairly by dividing each one into 5 pieces and giving each person 1 piece from each brownie. Each person receives $\frac{3}{5}$ of a brownie, as illustrated below by shaded portions.



Now let's apply the method for determining Egyptian fractions. If we begin with the same 3 brownies and find the largest unit fraction of a brownie that we can give to each of the 5 people, we could give each person $\frac{1}{2}$ brownie and still have $\frac{1}{2}$ brownie left to share.



12. Divide the last $\frac{1}{2}$ brownie into 5 equal pieces and share them evenly among 5 people. What fraction of a brownie will each person receive? How do you know?

13. Using this process as a guide, determine an Egyptian fraction representation for $\frac{3}{5}$.

14. Explain how the method for determining Egyptian fractions could be used to share 4 pizzas fairly among 5 people. Represent your solution and strategy with both a drawing and a written description.

In the brownie example, we saw that $\frac{1}{2}$ brownie results in a bigger piece than $\frac{1}{5}$ brownie. We can compare these fractions without finding common denominators. This is true for all unit fractions.

15. Explain how you know that $\frac{1}{2}$ brownie is greater than $\frac{1}{5}$ brownie. Describe how the strategy for comparing fractions can be applied to all unit fractions.

16. Explain how to use this strategy to compare $\frac{4}{9}$ and $\frac{4}{11}$.

17. Now imagine 2 pizzas of the same size. One has $\frac{7}{8}$ remaining, and the other has $\frac{11}{12}$ remaining. How can this fraction comparison strategy help us to determine which pizza has more remaining?

18. Use this strategy to compare $\frac{11}{13}$ and $\frac{17}{19}$. Explain your reasoning.

We can use Egyptian fractions to compare fractions. For example, consider $\frac{5}{8}$ and $\frac{6}{10}$. Both fractions are greater than $\frac{1}{2}$, so the benchmark of $\frac{1}{2}$ strategy is not sufficient for determining which fraction is greater. The numerators are not the same, and the same number of pieces are not "missing" from the fractions, so neither of these strategies will work on their own, either. Let's see how finding the Egyptian fraction representation might help.

With Egyptian fractions: $\frac{5}{8} = \frac{1}{2} + \frac{1}{8}$, and $\frac{6}{10} = \frac{1}{2} + \frac{1}{10}$. Both fractions are greater than $\frac{1}{2}$, so we must compare the parts that are greater than $\frac{1}{2}$. Since $\frac{1}{8}$ is greater than $\frac{1}{10}$, we can conclude that $\frac{5}{8}$ is greater than $\frac{6}{10}$. We just used a combination of strategies to determine which fraction is greater.

19. Use Egyptian fractions to compare $\frac{8}{15}$ and $\frac{6}{11}$. Explain your reasoning.

20. When ordering a list of fractions, it is often helpful to apply several comparison strategies. Order the following list of fractions from least to greatest:

$\frac{11}{13}$ $\frac{3}{5}$ $\frac{3}{7}$ $\frac{8}{14}$ $\frac{9}{16}$ $\frac{8}{10}$.

Describe the strategies you use to complete this task.

Can You …

- compare $\frac{22}{23}$ and $\frac{26}{27}$?
- determine the unit fraction sum for $\frac{2}{29}$?
- find three different unit fraction sums for $\frac{7}{8}$?

Did You Know That …

- Fibonacci proved that every simple fraction can be represented as the sum of unit fractions? The method for finding the sum is called a *greedy algorithm*.
- an infinite number of unit fraction sum representations exist for every simple fraction?

Mathematical Content

Representing fractions, comparing and ordering fractions, adding and subtracting fractions

References

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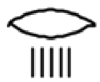
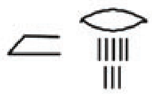
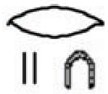
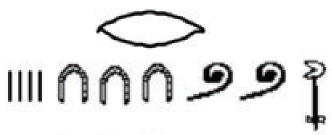
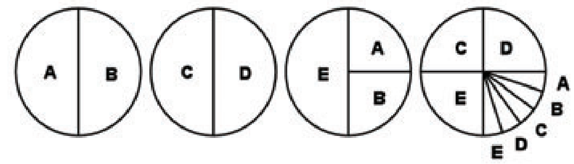
Egyptian Fractions—*Continued*

Answers

1. 1/4
2. 
3. Possible answer: Since 5/8 = 4/8 + 1/8, and 4/8 = 1/2, then 5/8 = 1/2 + 1/8.
4. Possible answers: 7/12 or 14/24.
5. Possible answer: 
6. 
7. 
8. Possible answer: The fraction 3/8 is less than 1/2. I know this because 4 is half of 8, so 4/8 is equal to 1/2. Since 3/8 is less than 4/8, then 3/8 must be less than 1/2.
9. The fraction 5/9 is greater than 3/7. I know this because 3 is less than half of 7, so 3/7 is less than 1/2; and 5 is greater than half of 9, so 5/9 is more than 1/2. Since 5/9 is greater than 1/2, and 3/7 is less than 1/2, 5/9 is greater than 3/7.
10. Possible answer: 5/9 = 1/2 + 1/18.
11. Possible answer: 3/4 = 2/3 + 1/12 and 3/4 = 1/2 + 1/4.
12. Each person receives an additional 1/10 brownie. I know this because if each person is given 1/2 brownie, then 5 half-size pieces will be shared, leaving 1/2 brownie. If that 1/2 brownie is cut into 5 equal pieces to share, each piece will actually represent 1/10 of a brownie (1/5 of 1/2 brownie).
13. Possible answer: 3/5 = 1/2 + 1/10.
14. I labeled my people A, B, C, D, and E. I give each person 1/2 pizza, and then I divide the remaining pizza into fourths. I give each person 1/4 pizza by labeling 5 of the fourths with the labels A, B, C, D, and E. I have 1/4 pizza left. If I divide that 1/4 into 5 equal pieces to share among the 5 people, and a whole pizza is 4/4, I know that each of those small sections is 1/20 of a pizza. Therefore, each person gets 1/2 + 1/4 + 1/20.



15. I know that 1/2 is greater than 1/5 because if I divide a whole brownie into 2 equal pieces, those pieces would be larger than the same-size whole brownie divided into 5 equal pieces. Since I am only comparing 1 piece in each case, I know that 1 half-size piece is greater than 1 fifth-size piece. This works with all unit fractions because we will always be comparing 1 piece to 1 other piece.
16. This strategy can be used to compare 4/9 and 4/11 because in this case we are comparing 4 pieces for both fractions. If we compare the same number of pieces, we need only examine the size of the pieces. Since ninths are larger than elevenths, we know that 4/9 is greater than 4/11.
17. Both pizzas are missing exactly 1 piece. If we compare the size of the "missing pieces," then we will know which pizza has more remaining. In this case, 1 pizza is missing 1/8, and the other is missing 1/12. We know that twelfths are smaller than eighths. Since we are comparing 1 piece of each, we know that 11/12 is missing a smaller piece of pizza. If 11/12 is missing a smaller piece of pizza, then it has more pizza remaining. So, 11/12 is greater than 7/8.
18. We know that 11/13 is 2/13 from the whole, and 17/19 is 2/19 from the whole. Since 2/13 is greater than 2/19, we conclude that 11/13 is farther from the whole, so 11/13 is less than 17/19.
19. Using Egyptian fractions, we see that 8/15 = 1/2 + 1/30 and 6/11 = 1/2 + 1/22. If 1/30 is less than 1/22, then 8/15 is less than 6/11.
20. The fraction order from least to greatest is 3/7, 9/16, 8/14, 3/5, 8/10, and 11/13 using the following strategies:
    a. I knew that 3/7 was the only fraction less than 1/2, so it was the first fraction that I listed.
    b. I used Egyptian fractions to represent 3/5 as 1/2 + 1/10, 8/14 as 1/2 + 1/14, and 9/16 as 1/2 + 1/16, so I only had to compare 1/10, 1/14, and 1/16 to order them.
    c. I noticed that 3/5, 11/13, and 8/10 were each missing 2 pieces from their wholes, so I compared 2/5, 2/13, and 2/10. The greater the size of these fractions, the smaller the size of the original fraction. Since 3/5 was the smallest of these 3 fractions, I knew that the entire list was in order.

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